1 a) $\mathcal{V} = \mathbb{R}^2$, $\mathcal{U} = \operatorname{span}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$, $\mathcal{W} = \operatorname{span}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$.

Both $\mathcal{U}$ and $\mathcal{W}$ are subspaces of $\mathcal{V}$.

Now take $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \in \mathcal{U} \cup \mathcal{W}$.

Since $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \notin \mathcal{U} \cup \mathcal{W}$,

$\mathcal{U} \cup \mathcal{W}$ is not a subspace of $\mathcal{V}$.

b) If $\mathcal{U} \subseteq \mathcal{W}$ then $\mathcal{U} \cup \mathcal{W} = \mathcal{W}$ so $\mathcal{U} \cup \mathcal{W}$ is a subspace.

Similarly, if $\mathcal{W} \subseteq \mathcal{U}$ then $\mathcal{U} \cup \mathcal{W} = \mathcal{U}$ so $\mathcal{U} \cup \mathcal{W}$ is a subspace.

c) Assume that $\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$.

Suppose that there exists a vector $u \in \mathcal{U}$ s.t. $u \notin \mathcal{W}$.

Let $w \in \mathcal{W}$. Then $w \in \mathcal{U} \cup \mathcal{W}$. Also, $u \in \mathcal{U} \cup \mathcal{W}$.

Since $\mathcal{U} \cup \mathcal{W}$ is a subspace, $u + w \in \mathcal{U} \cup \mathcal{W}$.

We conclude that $u + w \in \mathcal{U}$ or $u + w \in \mathcal{W}$.

d) Assume that $\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$.
If $\mathcal{U} \subseteq \mathcal{W}$ then we are done.
Suppose $\mathcal{U} \nsubseteq \mathcal{W}$. Then there exists a $u \in \mathcal{U}$ such that $u \notin \mathcal{W}$.
Let $w \in \mathcal{W}$. By c), either $u+w \in \mathcal{U}$ or $u+w \in \mathcal{W}$.

If $u+w \in \mathcal{W}$ then $\underbrace{u+w}_{\in \mathcal{W}} \underbrace{-w}_{\in \mathcal{W}} = u \in \mathcal{W}$ since $\mathcal{W}$ is a subspace. This, however, is a contradiction by .

We must therefore have that $u+w \in \mathcal{U}$.
But then $\underbrace{u+w}_{\in \mathcal{U}} \underbrace{-u}_{\in \mathcal{U}} = w \in \mathcal{U}$, since $\mathcal{U}$ is a subspace. Because $w \in \mathcal{W}$ was arbitrary, we conclude that $\mathcal{W} \subseteq \mathcal{U}$. $\square$

2 a) $T : P_2 \to P_2$ is defined by

$$(T(p))(x) = x^2 p\left(\frac{1}{x}\right) + p''(x).$$

Let $p, q \in P_2$ and $a, b \in \mathbb{R}$. To show:
$T(ap+bq) = aT(p) + bT(q)$.

Write $p(x) = c_0 + c_1 x + c_2 x^2$
$q(x) = d_0 + d_1 x + d_2 x^2$
where $c_i, d_i \in \mathbb{R}$ for $i = 0, 1, 2$.

$$T(p) = x^2\left(c_0 + c_1 \frac{1}{x} + c_2 \frac{1}{x^2}\right) + 2c_2$$
$$= 3c_2 + c_1 x + c_0 x^2.$$

Similarly,
$$T(q) = 3d_2 + d_1 x + d_0 x^2$$

Now, compute $T(ap+bq)$
$$= T\left(a(c_0 + c_1 x + c_2 x^2) + b(d_0 + d_1 x + d_2 x^2)\right)$$
$$= T\left(ac_0 + bd_0 + (ac_1 + bd_1)x + (ac_2 + bd_2)x^2\right)$$
$$= 3(ac_2 + bd_2) + (ac_1 + bd_1)x + (ac_0 + bd_0)x^2$$
$$= a(3c_2 + c_1 x + c_0 x^2) + b(3d_2 + d_1 x + d_0 x^2)$$
$$= aT(p) + bT(q).$$

So T is a linear operator.

b) $T(1) = x^2 = -3 \cdot 1 + 2(1+x) + 1 \cdot (1-2x+x^2)$

$T(1+x) = x^2\left(1+\frac{1}{x}\right) = x + x^2$

$= -4 \cdot 1 + 3 \cdot (1+x) + 1 \cdot (1-2x+x^2)$

$T(1-2x+x^2) = x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right) + 2$

$= 3 - 2x + x^2$

$= 2 \cdot 1 + 0 \cdot (1+x) + 1 \cdot (1-2x+x^2)$

So the matrix representation of T with respect to $(1, 1+x, 1-2x+x^2)$ is:

$$\begin{pmatrix} -3 & -4 & 2 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

c) R maps the polynomial x^2 to $\frac{1}{x} + 2$, which is not in P_2. So R is not a mapping from P_2 to itself and therefore not a linear operator on P_2.

$S(x^2) = 4$, while $S(x^2 + x^2) = 32$.

So $S(x^2+x^2) \neq S(x^2) + S(x^2)$.

Thus, S is not a linear operator.

3 a) $A>0$ means that $x^T A x > 0$ for all nonzero vectors $x \in \mathbb{R}^n$. Let $k \in \{1, 2, \dots, n\}$.
Also let $y \in \mathbb{R}^k$ be an arbitrary nonzero vector.

$$0 < \begin{pmatrix} y \\ 0 \end{pmatrix}^T A \begin{pmatrix} y \\ 0 \end{pmatrix} = y^T A_k y.$$

So $A_k > 0$.

During the lectures, we have seen that $A_k > 0$ if and only if all eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ of A_k are positive.
Thus, $\det(A_k) = \lambda_1 \cdot \lambda_2 \cdots \lambda_k > 0$.

b) $A_i > 0$ implies that all eigenvalues of A_i are positive.
Since the eigenvalues of

$$P := \begin{bmatrix} A_i & 0 \\ 0 & a_i - b_i^T A_i^{-1} b_i \end{bmatrix}$$

are the union of the eigenvalues of A_i and $a_i - b_i^T A^{-1} b_i > 0$, the eigenvalues of P are positive. Thus, P is positive definite.

Now, note that

$$M = \begin{bmatrix} I & A_i^{-1} b_i \\ 0 & 1 \end{bmatrix}$$

is nonsingular (it has determinant 1).

Therefore, if $x \in \mathbb{R}^n$ is nonzero, also $Mx \neq 0$.

In addition,

$$M^T = \begin{pmatrix} I & 0 \\ b_i^T A_i^{-1} & 1 \end{pmatrix},$$ since A_i is symmetric.

Hence, for any nonzero $x \in \mathbb{R}^n$, $x^T A_{i+1} x =$ $x^T M^T P \underbrace{Mx}_{\neq 0} > 0$, by positive definiteness of P.

As such, $A_{i+1} > 0$.

c) Assume that $\det(A_k) > 0$ for $k = 1, 2, \dots, n$.

We will prove that $A_k > 0$ for $k = 1, 2, \dots, n$, thus $A = A_n > 0$.

① $A_1 = \det(A_1) > 0$ ✓

② Assume that $A_i > 0$ for some $i \in \{1, 2, \ldots, n-1\}$.
To prove: $A_{i+1} > 0$.

$$A_{i+1} = M^T \begin{bmatrix} A_i & 0 \\ 0 & a_i - b_i^T A_i^{-1} b_i \end{bmatrix} M.$$

$$\det(A_{i+1}) = \underbrace{\det(M^T)}_{=1} \cdot \det(P) \cdot \underbrace{\det(M)}_{=1}$$

while $\det(P) = \det(A_i) \cdot \det(a_i - b_i^T A_i^{-1} b_i)$

so $\underbrace{\det(A_{i+1})}_{>0} = \underbrace{\det(A_i)}_{>0} \cdot \det(a_i - b_i^T A_i^{-1} b_i)$.

$\Rightarrow \det(a_i - b_i^T A_i^{-1} b_i) = a_i - b_i^T A_i^{-1} b_i > 0$
$\Rightarrow A_{i+1} > 0$ by (b).

Therefore, by induction, $A_k > 0$ for $k = 1, 2, \ldots, n$.
Hence, $A = A_n > 0$.

d) $1 > 0$, $\det \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = 2 > 0$

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 1 > 0.$$

So A is positive definite by (c).

4 a)

$$A = \begin{bmatrix} a & -b \\ b & a \\ c & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} a^2+b^2+c^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$$

so eigenvalues of A^TA are $a^2+b^2+c^2 > a^2+b^2$.

$\Rightarrow \sigma_1 = \sqrt{a^2+b^2+c^2}$ and $\sigma_2 = \sqrt{a^2+b^2}$.

note that $\sigma_1 > 0$ and $\sigma_2 > 0$, since $a, b, c \neq 0$.

A corresponding orthogonal matrix V is:

$$V = \begin{array}{c} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{matrix} v_1 & v_2 \end{matrix} \end{array}.$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$$

Extend u_1, u_2 to an orthonormal basis of $\mathbb{R}^3$:

For example, choose $u_3 = \dfrac{1}{\sqrt{a^2+b^2+\frac{1}{c^2}(a^2+b^2)^2}} \begin{pmatrix} a \\ b \\ \frac{1}{c}(-a^2-b^2) \end{pmatrix}$

Define $U = \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2+c^2}} & \frac{-b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2+\frac{1}{c^2}(a^2+b^2)^2}} \\ \frac{b}{\sqrt{a^2+b^2+c^2}} & \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2+\frac{1}{c^2}(a^2+b^2)^2}} \\ \frac{c}{\sqrt{a^2+b^2+c^2}} & 0 & \frac{-a^2-b^2}{c\sqrt{a^2+b^2+\frac{1}{c^2}(a^2+b^2)^2}} \end{pmatrix}$

By construction, U is an orthogonal matrix.

Define $\Sigma = \begin{pmatrix} \sqrt{a^2+b^2+c^2} & 0 \\ 0 & \sqrt{a^2+b^2} \\ 0 & 0 \end{pmatrix}$

Then $A = U\Sigma V^T$.

b) A best rank-1 approximation is

$$X = U \begin{pmatrix} \sqrt{a^2+b^2+c^2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} V^T$$

$$= \begin{pmatrix} a & 0 \\ b & 0 \\ c & 0 \end{pmatrix}$$

The distance of A to the set of 3×2 matrices of rank ≤ 1 is:

$$d(A, M_1) = \sigma_2 = \sqrt{a^2+b^2}$$